June, 2021: Vector spaces, basis, dimension.
SWMS-Worksheet-II in Linear Algebra

Notation: F denotes the set of real numbers $\mathbb{R}$, or the set of complex numbers $\mathbb{C}$.

(1) Determine whether the set of vectors

$$S = \{v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}\}$$

in $\mathbb{R}^3$ is a linearly independent set.

(2) Show that the column vectors of a square matrix A are linearly independent if and only if the linear system $AX = 0$ has the only zero solution.

(3) For which real numbers x do the vectors $(x, 1, 1, 1)^t$, $(1, x, 1, 1)^t$, $(1, 1, x, 1)^t$, $(1, 1, 1, x)^t$ NOT form a basis of $\mathbb{R}^4$? For each of the values of x that you find, what is the dimension of the subspace of $\mathbb{R}^4$ that they span?

(4) Show that the subset W of all symmetric $n \times n$ matrices forms a subspace of $M_n(F)$, the vector space of all $n \times n$ matrices. Find a basis for the subspace W of $M_n(F)$.

(5) (i) Find the dimension of the subspace which is the intersection of the following two planes in $\mathbb{R}^3$

$$x + 2y - z = 0, \qquad 3x - 3y + z = 0.$$

(ii) Can you write the above subspace as the nullspace (or kernel) of a linear transformation?

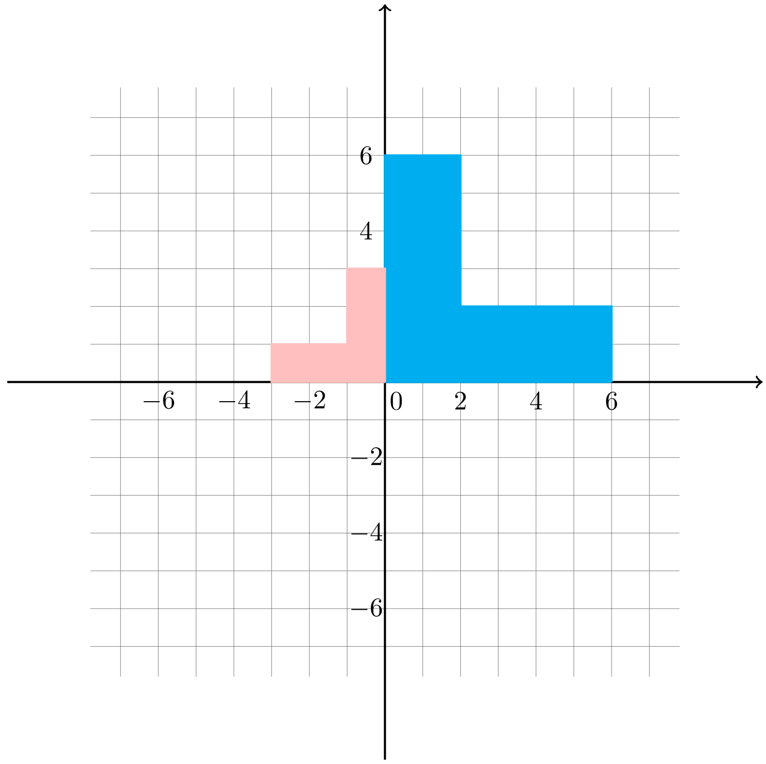
(6) Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 3 & -1 & 1 & 7 & 0 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}.$$

(7) Find a basis for the column space of

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 2 & 4 & -1 & 1 & 0 \\ 3 & 6 & -1 & 4 & 1 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}.$$

(8) Find a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which maps the region **L** in blue to the region **L** in pink.



(9) Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$T(x, y, z)^t = (x + 2y - z, 2x + 3y + z, 4x + 7y - z)^t.$$

Describe the null space of T and the range of T. Geometrically what does these two subspaces of $\mathbb{R}^3$ represent?

(10) Let $\mathbb{R}^3 \to \mathbb{R}^3$ be the linear map T_A given by multiplication by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find the matrix of T_A with respect to the basis $\mathcal{B} = (1\ 1\ 1)^t, (1\ 1\ 0)^t, (1\ 0\ 0)^t$. Find range of the linear transformation.