SWMS 2021

A Short Note on Linear Programming

Convex polytopes and polyhedral cones appear very naturally in problems of linear optimization.

Definition. A **linear program in standard form** is any problem of the following form:

$$\begin{array}{ll} \text{Find a vector} & v = (v_1, ..., v_n) \in \mathbb{R}^n \\ \text{that minimizes} & f(v) = c_1 v_1 + \cdots + c_n v_n \\ \text{subject to} & Av \le b \quad \text{and} \quad v \ge 0, \end{array}$$

where A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$.

The function f is called the **objective function**.

The region $P = \{v \in \mathbb{R}^n : Av \le b,\ v \ge 0\}$, if nonempty, is called the **feasible region** of the problem. If P is empty, the problem is called **infeasible**.

Examples (and nonexamples). 1. What is the largest rectangle (in terms of area) of perimeter at most 2?

2. A farmer wants his cow to be as skinny as possible while still keeping her healthy. There are 2 different food types available: A and B, and the cow needs certain quantities of three vitamins: I, II and III. Suppose food A contains c_1 calories per kg, and a_1, a_2 and a_3 mgs per kg of vitamins I, II and III, respectively. Suppose B contains c_2 calories per kg, and b_1, b_2 and b_3 mgs per kg of vitamins I, II and III, respectively. The cow requires at least m_1, m_2 and m_3 mgs of I, II and III, respectively, to stay healthy. Given that the goal is to minimize caloric intake while having enough of each vitamin, how should she be fed?

3. There are 168 hours in a week. Say we want to allocate our time between classes and studying (S), fun activities (F), and everything else (E) (eating, sleeping, taking showers, etc). Suppose that to survive we need to spend at least 56 hours on E (8 hours/day). To maintain sanity we need $F + E \ge 70$. To pass our courses, we need $S \ge 60$, but more if we don't sleep enough or spend too much time having fun: $2S + E - 3F = 150$. (E.g., if we spend more time on F then we need to sleep more or study more). Suppose our notion of happiness is expressed by $2F + E$. How can you maximize your happiness?