LECTURE - 1

[1]

Matrix operations

Let m, n be positive integers. An $m \times n$ matrix is a collection of mn numbers arranged in a rectangular array

$$\left.\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\right\} m \text{ rows}$$

$$\underbrace{\hphantom{a_{11} \quad a_{12} \quad \cdots \quad a_{1n}}}_{n \text{ columns}}$$

A matrix is usually denoted by capital letters, and the entries of the matrix A are denoted by a_{ij}, where i and j are indices, $1 \le i \le m$, $1 \le j \le n$, i denotes the row and j denotes the column. So, a_{ij} is the number appearing at the ith row and the jth column of A.

An $n \times n$ matrix is called a square matrix, a $n \times 1$ matrix is called a column vector, whereas a $1 \times n$ matrix is called a row vector, for n an integer greater than 1.

$[1 \quad 3 \quad -2]$ or $(1 \quad 3 \quad -2)$ is a row vector,

and $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ is a column vector.

Addition of matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. We define their sum to be the $m \times n$ matrix $A+B$, such that the (i,j)th entry of $A+B$ is $a_{ij} + b_{ij}$, for all $1 \le i \le m$, and $1 \le j \le n$.

Example: $\begin{bmatrix} 0 & 3 & 2 \\ 1/2 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 3 & 3 \\ 5/2 & -2 & 7 \end{bmatrix}$

Scalar multiplication of a matrix by a number.

Let A be a $m \times n$ matrix and c be a number. Then the scalar multiplication of A by c, denoted by cA, is the matrix whose (i,j)th entry is ca_{ij}, $\forall\ 1 \le i \le m, 1 \le j \le n$.

Example: $\frac{1}{2}\begin{bmatrix} 3 & -1 \\ 5 & 11 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ 5/2 & 11/2 \end{bmatrix}$

Matrix multiplication

How do we multiply a row-vector and a column vector? Let $A = (a_{11}\ a_{12} \cdots a_{1n})$ be a row vector, and $B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{1n} \end{bmatrix}$ be a column vector. Note that they are of same dimension. Then the product ~~of~~ AB is a number, which is $\sum_{i=1}^{n} a_{1i} b_{i1}$. AB in this case is a 1×1 matrix.

Example : $\begin{pmatrix} \frac{1}{5} & 1 & 3 \end{pmatrix} \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = \frac{1}{5} \cdot 1 + 1 \cdot (-2) + 3 \cdot 5$

$= \frac{1}{5} - 2 + 15$

$= \frac{66}{5}$.

Product of two matrices A and B are defined when the number of columns of A are same as the number of rows of B.

Let A be a $m \times l$ matrix and B be a $l \times n$ matrix. Then the product AB is a $m \times n$ matrix, whose (i,j)th entry is the product of the ith row of A and the jth column of B. Hence, if the (i,j)th entry of AB is denoted by s_{ij}, then

$$s_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{il} b_{lj}$$

Example :

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix}_{2 \times 3}$, $B = \begin{bmatrix} 0 & -1 & -2 \\ 1 & -5 & 7 \\ 1 & 4 & 1 \end{bmatrix}_{3 \times 3}$,

then $AB = \begin{bmatrix} 1 & -15 & 11 \\ 8 & -13 & 38 \end{bmatrix}_{2 \times 3}$

[2] SYSTEM OF EQUATIONS

Let us considere the following ~~equations~~ system of m equations in n unknowns :

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m . \end{aligned}\right\} \mathrm{I}$$

Here $x_1, x_2, \ldots, x_n$ are the n variables, and a_{ij}, $1 \le i \le m$, $1 \le j \le n$, and b_i, $1 \le i \le m$ are constants.

This system of equations (I) can be written in matrix notation as

$$AX = B,$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

are matrices whose entries are constants and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the variable column vector.

We shall deduce ways to solve this system of equations, using matrix operations.

Verify that, matrix operations satisfy the following laws, whenever sizes are suitable:

1. $A(B+B') = AB + AB'$
2. $(A+A')B = AB + A'B$
3. $(AB)C = A(BC)$
4. $c(AB) = (cA)B = A(cB)$.

But, ~~in general~~, multiplication of matrices, is not ~~generally~~ commutative in general, ie. $AB \neq BA$ in general, for square matrices A and B.

Definitions

1. Zero matrix : A matrix whose all entries are zeroes.
2. Diagonal matrix : A square matrix D is a diagonal matrix if its only non-zero entries are diagonal entries, ie a_{ii}'s are possibly non-zeroes, and $a_{ij} = 0$, $\forall\ i \neq j$.
3. Identity matrix : A square matrix whose diagonal entries are 1's and off-diagonal entries are all zeroes. If of order $n \times n$, it is denoted by I_n.
4. Upper triangular matrix : A square matrix A such that $a_{ij} = 0 \ \forall\ i > j$.
5. Lower triangular matrix : A square matrix A such that $a_{ij} = 0 \ \forall\ i < j$.

6. <u>Invertible matrix</u> : Let A be a square $n \times n$ matrix. If there is a matrix B such that $AB = I_n$ and $BA = I_n$, then B is called an <u>inverse of A</u> and is denoted by A^{-1}.
In this case, A is called an invertible matrix.

<u>Lemma 1.1</u>: Let A be a square matrix that has a right inverse, ie a matrix B such that $AB = I$ and also a left inverse, ie C such that $CA = I$. Then $B = C$.

<u>Proof</u> :
$$B = IB = (CA)B$$
$$= C(AB)$$
$$= CI = C.$$

<u>Proposition 1.2</u>

ⓐ Let A be an invertible matrix, then the inverse of A is also invertible.

<u>Proof</u> : Inverse of A^{-1} is A, as is shown here : $AA^{-1} = A^{-1}A = I_n$.

ⓑ Let A, B be $n \times n$ matrices such that A and B are both invertible. Then the product AB is also invertible.

<u>Proof</u> : $(AB)^{-1} = B^{-1}A^{-1}$, as

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = I_n,$$

and $B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = I_n$.

(c) If $A_1, \ldots, A_m$ are all $n \times n$ matrices, which are invertible, then the product $A_1 \cdots A_m$ is also invertible, and

$$(A_1 \cdots A_m)^{-1} = A_m^{-1} \cdots A_1^{-1}$$

<u>Proof</u>: We ~~verified~~ proved the case for $m=2$. Let this statement be true for $m-1$. We shall prove it for m. (This is called proving by mathematical induction).

Let $C = A_1 \cdots A_{m-1}$.

So $A_1 \cdots A_{m-1} A_m = C A_m$

By part (b), $(CA_m)^{-1} = A_m^{-1} C^{-1}$

$$= A_m^{-1} (A_1 \cdots A_{m-1})^{-1}$$

$$= A_m^{-1} A_{m-1}^{-1} \cdots A_1^{-1}$$

(by induction hypothesis, ie. by our assumption).

Hence, proved.

<u>Exercise 1</u>: Find the inverse of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if it exists? When does it exist?

<u>Soln</u>: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} B = B \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2$

gives $B = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. So the inverse exists if and only if $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ is non-zero.

Exercise 2. Show that a square matrix that has either a row of zeroes or a column of zeroes is not invertible.

Matrix units

These are the simplest non-zero matrices. A $m \times n$ matrix unit has one non-zero entry, which is 1, at the (i,j)-th place, say, and all other entries 0. Such a matrix is denoted by e_{ij}.

Note that, every $m \times n$ matrix can be written as follows:

$$A = a_{11}e_{11} + a_{12}e_{12} + \cdots + a_{mn}e_{mn}$$

$$= \sum_{i,j} a_{ij}e_{ij} \quad \text{(addition of matrices)}$$

Here, $a_{ij}e_{ij}$ is the scalar multiplication of the matrix e_{ij} by the scalar a_{ij}.

Exercise 3: Show that if there exist numbers $c_{11}, c_{12}, \ldots, c_{mn}$ ($m.n$ of them) such that

$$c_{11}e_{11} + c_{12}e_{12} + \cdots + c_{ij}e_{ij} + \cdots + c_{mn}e_{mn}$$

is the zero matrix, then

$$c_{11} = c_{12} = \cdots = c_{ij} = \cdots = c_{mn} = 0$$

[3] SOLVING SYSTEM OF LINEAR EQNS.

Let us first introduce elementary matrices.

Elementary matrices

Type I :

$$\begin{array}{cc} & \begin{matrix} & & & j & \end{matrix} \\ i & \begin{bmatrix} 1 & & & & \\ & 1 & & a & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}_{n\times n} \end{array} \quad \text{or} \quad \begin{array}{cc} & \begin{matrix} j & & & \end{matrix} \\ i & \begin{bmatrix} 1 & & & \\ & 1 & & \\ a & & 1 & \\ & & & 1 \end{bmatrix}_{n\times n} \end{array} \qquad (i \neq j)$$

All diagonal entries are 1's. There is only one non-zero off diagonal entry.

These can be represented by $I_n + a e_{ij}$, $a \neq 0$.

Eg.: $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = I_3 + 2e_{23}$$

Type II :

$$\begin{array}{cc} & \begin{matrix} & i & & j & \end{matrix} \\ \begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} & \begin{bmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{bmatrix} \end{array}$$

The ith and jth diagonal entries of the identity matrix are replaced by zero, and 1's are added to (i,j)th and (j,i)th positions. All other entries are zeros.

These elementary matrices look like

$$I_n - e_{ii} - e_{jj} + e_{ij} + e_{ji}, \quad i \neq j$$

<u>Type III</u> :

$$\begin{array}{c c} & \begin{array}{ccccc} & & i & & \end{array} \\ i & \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & c & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \end{array} \qquad (c \neq 0)$$

Here one diagonal entry of the identity matrix is replaced by a non-zero scalar c.
These are represented by $I_n + (c-1)e_{ii}$

<u>Examples</u>

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Type II Type III

What happens when you multiply a $n \times n$ matrix A by an elementary $n \times n$ matrix E, from the left ?

If <u>E is of type I</u>, the matrix EA is same as the matrix obtained from A by replacing the ith row of A by ith row of A + a.(jth row of A), where $E = I_n + a e_{ij}$.

(2,3)-th place

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 5 & 3 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{EA}{\begin{bmatrix} 1 & 0 & 2 \\ 1 & 7 & 5 \\ 1 & 1 & 1 \end{bmatrix}}$$

→ row 2 of A is replaced by row 2 + 2.(row 3)

<u>If E is of type II</u> : The matrix EA is same as interchanging row i and row j of A, if $E = I_n + e_{ij} + e_{ji} - e_{ii} - e_{jj}$

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} -1 & 2 & 1 & 3 & 5 \\ 0 & 1 & -2 & 4 & 0 \\ 3 & -2 & -3 & 1 & 0 \\ 1 & 4 & 2 & -4 & 1 \\ -2 & 7 & 1 & 11 & 3 \end{bmatrix}} = \underset{EA}{\begin{bmatrix} -1 & 2 & 1 & 3 & 5 \\ 0 & 1 & -2 & 4 & 0 \\ -2 & 7 & 1 & 11 & 3 \\ 1 & 4 & 2 & -4 & 1 \\ 3 & -2 & -3 & 1 & 0 \end{bmatrix}}$$

In the resultant matrix EA, the third row of A is swapped with the fifth row of A.

<u>If E is of type III</u> : The matrix EA is same as multiplying the ith row of A by the non-zero scalar c, where $E = I_n + (c-1)e_{ii}$.

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 & -1 \\ 7 & 2 & 5 \\ -2 & 3 & 1 \end{bmatrix}} = \underset{EA}{\begin{bmatrix} 1 & 2 & -1 \\ 7 & -2 & -5 \\ -2 & 3 & 1 \end{bmatrix}}$$

In EA, the 2nd row of A is multiplied by −1.

Elementary row operations on A :

(a) Add a.(row j) to row i ; can be brought about by multiplying A by an elm. matrix of type I, from left,

(b) Interchange (row i) and (row j) ; can be brought about by multiplying A by a type II elm. matrix, from left,

(c) Multiply (row i) by a non-zero scalar c; can be brought about by multiplying A by a Type III elm. matrix, from left.

Exercise 4.

Elementary matrices are invertible.

Row reduction :

The process of simplifying a matrix by performing a sequence of row operations, or equivalently, multiplying the matrix by a sequence of elementary matrices from the left, is called row reduction.

So, if M is a matrix, a row reduced matrix of M is M', where M' is given by

$$M' = E_k \cdots E_2 E_1 M,$$

where $E_1, \ldots, E_k$ are elementary matrices.

Row reduction is used to solve systems of linear equations.

Suppose we are given a system of m equations in n unknowns, say $AX = B$, where A is a $m \times n$ matrix, ~~and~~ B is a column vector, and X is an unknown column vector.

To solve this system, we form the $m \times (n+1)$ block matrix, also called augmented matrix

$$M = [A|B] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_n \end{array}\right]$$

and do row operations to simplify M.

Let $M' = E_k \cdots E_2 E_1 M = [E_k \cdots E_1 A \,|\, E_k \cdots E_1 B]$

$$= [A' \,|\, B'], \text{ say.}$$

We prove that :

<u>Proposition 1.3</u> The systems $A'x = B'$ and $AX = B$ have the same solutions.

~~Let~~ <u>Proof</u> : Let $P = E_k \cdots E_1$.

$M' = PM = [PA \,|\, PB] = [A' \,|\, B']$, where $A' = PA$, $B' = PB$.

If x_0 is a solution of $Ax = B$, ie $AX_0 = B$, then $PAX_0 = PB$, ie $A'x_0 = B'$. Thus, x_0 is also a solution of $A'x = B'$.

Conversely, let x_1 be a solution of $A'x = B'$. Note that P being a product of invertible matrices, is itself invertible.

$$A'x_1 = B'$$

$$\Rightarrow P^{-1}A'x_1 = P^{-1}B'$$

$$\Rightarrow AX_1 = B.$$

Thus, x_1 is a solution of $AX = B$ too. $\square$

A row echelon matrix is one that has these properties :

1. If the ith row is zero, then every row below ith row is also a zero row.
2. If ~~row~~ ith row is non-zero, then first non-zero entry of this ith row is 1. It is called the pivot of the ith row.
3. If row$(i+1)$ is non-zero, then the pivot of the $(i+1)$th row is to the right of the pivot of row(i).
4. The entries above a pivot are zero. (Entries below a pivot are anyway zeroes by 3).

Algorithm to reduce a matrix M to its row reduced echelon form :

Assume M is a non-zero matrix.

i) Find the first column that contains a non-zero entry, say a.

ii) Interchange rows (by Type II elm. matrices) to move a to the first row.

iii) Multiply first row by $\frac{1}{a}$, (Type III), So that a in first row is replaced by 1. This is the pivot of 1st row.

iv) All entries below this pivot can be made zeroes by operations of type I (Type I elm matrices).

The resulting matrix now looks like

$$\left[\begin{array}{ccc|c|ccc} 0 & \cdots & 0 & 1 & * & \cdots & * \\ \hline 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & * & \cdots & * \end{array}\right] = \left[\begin{array}{c|c|c} & 1 & B_1 \\ \hline & & D_1 \end{array}\right]$$

v) Reduce D_1 to a row echelon form, say D_2.

vi) The entries in B_1, above the pivots in D_1 can be made into zeroes, to complete the row reduction process.

<u>Exercise 5</u>: Solve the system of equations:

$$\left.\begin{aligned} 2x_2 + 4x_3 &= 2 \\ 2x_1 + 4x_2 + 2x_3 &= 3 \\ 3x_1 + 3x_2 + x_3 &= 1 \end{aligned}\right\} \quad (*)$$

<u>Solution</u> : We construct the augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 2 \\ 2 & 4 & 2 & 3 \\ 3 & 3 & 1 & 1 \end{array}\right]$$

We perform row operations to reduce it to a row echelon form:

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & 2 \\ 2 & 4 & 2 & 3 \\ 3 & 3 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 4 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 3 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3/2 \\ 0 & 2 & 4 & 2 \\ 3 & 3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3/2 \\ 0 & 2 & 4 & 2 \\ 0 & -3 & -2 & -7/2 \end{array}\right] \xrightarrow[-R_3]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3/2 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 2 & 7/2 \end{array}\right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3/2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3/2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & 1/2 \end{array}\right] \xrightarrow{-\frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3/2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1/8 \end{array}\right] \xrightarrow{R_1 - 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -1/2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1/8 \end{array}\right] \xrightarrow[R_2 - 2R_3]{R_1 + 3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7/8 \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & -1/8 \end{array}\right]$$

Reading out the equation at this stage, we get

$$x_1 = -7/8$$
$$x_2 = 5/4$$
$$x_3 = -1/8.$$

Thus, by proposition 1.3, ~~we~~ $X = \begin{bmatrix} -7/8 \\ 5/4 \\ -1/8 \end{bmatrix}$ is a solution (and the unique one) of the system of equation (*).

Exercise 6

Find solution of the following system of equations in three variables using row-reduction.

$$x_1 + 2x_2 - 3x_3 = -2$$
$$3x_1 - x_2 - 2x_3 = 1$$
$$2x_1 + 3x_2 - 5x_3 = -3$$

Solution : Here $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right]$$

Row reducing $[A|B]$ we get the row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the equations read:

$$x_1 \quad - x_3 = 0$$
$$x_2 - x_3 = -1.$$

Let $x_3 = c$. Then $x_1 = x_3 = c$, and $x_2 = x_3 - 1 = c - 1$.

Thus, any solution of the system is of the form

$$X = \begin{bmatrix} c \\ c \\ c-1 \end{bmatrix} \text{ or } X = c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \text{ for some scalar } c.$$

Thus, the system has infinitely many solutions.

Exercise 7: Find solution of the following system of equations in three variables using row-reduction.

$$x_1 + x_2 + x_3 = -1$$
$$3x_1 - x_2 - x_3 = 4$$
$$x_1 + 5x_2 + 5x_3 = -1$$

Solution: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right] = [A|B]$

The row reduced echelon matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 1 & -7/4 \\ 0 & 0 & 0 & 7/4 \end{array}\right]$

The third equation here would read like $0 = 7/4$, which is absurd. Thus, this system of equations has NO solution. Such a system is called ~~an~~ <u>inconsistent</u>.

The following proposition is straight forward.

<u>Proposition 1.4</u> : Let $M' = [A' \mid B']$ be a (augmented) row echelon matrix. The system of equations $A'X = B'$ has a solution if and only if there is no pivot in the last column B'.

In case, there is no pivot in the last column, arbitrary values are assigned to the variables x_i, provided column i does not contain a pivot. The other unknowns can then be determined uniquely, in terms of these arbitrary values, as in exercise 6.

<u>Corollary 1.5</u> : Every system $AX = 0$ of m homogeneous equations in n unknowns, with $m < n$, has a solution X in which some x_i is non-zero.

Theorem 1.6 : Let A be a square matrix. The following are equivalent.

ⓐ A can be reduced to the identity by a sequence of elementary row operations.

ⓑ A is a product of elementary matrices.

ⓒ A is invertible.

ⓓ The system of equations $AX = B$ has a unique solution for every column vector B.

ⓔ The system of homogeneous equations $AX = 0$ has only the trivial solution $X = 0$.

Proof : Let ⓐ hold. Then $\exists$ elm. matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A = I$.

$$\Rightarrow E_1^{-1} \cdots E_k^{-1} = A.$$

Thus, ⓑ holds.

ⓑ $\Rightarrow$ ⓒ is easy to see.

If ⓒ ~~had~~ holds, the row reduced echelon form $A' = E_k \cdots E_1 A$ is also invertible.

Now, a row reduced echelon square matrix is either identity, or has last row zero. Since a matrix with a zero row cannot be invertible, A' must be the identity matrix. Thus the system of equations $AX = B$ has a unique solution $X = B'$.

Thus, ⓒ $\Rightarrow$ ⓓ.

(d) $\Rightarrow$ (e) is clear, as $X=0$ is a solution to $AX=\bar{0}$.
If (a) is not true, then the last row of the row reduced echelon form of A is the zero row. Since A is square, there are lesser number of pivots in A' than n. Thus, we can assign arbitrary values to the variables x_i, where column i doesn't contain a pivot. Thus $A'X=0$ has non-trivial solutions, and so does $AX=0$.
Thus if (a) is not true, then (e) is also not true.
Thus, (e) $\Rightarrow$ (a). □.

[4] TO FIND INVERSE OF A SQUARE MATRIX USING ROW REDUCTIONS :

Let A be an invertible matrix, of order $n \times n$.
$\Leftrightarrow$ The row reduced echelon form of A is I_n.
So, $I_n = E_K E_{K-1} \cdots E_1 A$, for some elementary matrices $E_1, \ldots, E_K$.
This equation suggests that $A^{-1} = E_K E_{K-1} \cdots E_1 I_n$
To find the inverse of A, start with the block matrix $(n \times 2n)$

$$\left[\begin{array}{c|c} A & I_n \end{array}\right]_{n \times 2n}$$

, and apply row reductions. At the point where the left block turns to be identity, the right block gives you the inverse.

— x —