

$$\int_a^b P_2(x) dx = \int_a^b \left( \frac{(x-a)(x-\frac{a+b}{2})}{(b-a)(b-\frac{a+b}{2})} f(b) + \frac{(x-\frac{a+b}{2})(x-b)}{(a-\frac{a+b}{2})(a-b)} f(a) \right.$$

$$\left. + \frac{(x-a)(x-b)}{(\frac{a+b}{2}-a)(\frac{a+b}{2}-b)} f\left(\frac{a+b}{2}\right) \right) dx$$

$$= \frac{f(b)}{(b-a)(\frac{b-a}{2})} \int_a^b (x^2 - x(a+\frac{a+b}{2}) + a(\frac{a+b}{2})) dx$$

$$+ \frac{f(a)}{(\frac{a-b}{2})(a-b)} \int_a^b (x^2 - x(\frac{a+b}{2} + b) + b(\frac{a+b}{2})) dx$$

$$+ \frac{f(\frac{a+b}{2})}{(\frac{b-a}{2})(\frac{a-b}{2})} \int_a^b (x^2 - x(a+b) + ab) dx$$

$$= \frac{2f(b)}{(b-a)^2} \left[ \frac{x^3}{3} - \frac{x^2}{2} (3\frac{a+b}{2}) + a(\frac{a+b}{2})x \right]_a^b$$

$$+ \frac{2f(a)}{(a-b)^2} \left[ \frac{x^3}{3} - \frac{x^2}{2} (\frac{a+3b}{2}) + b(\frac{a+b}{2})x \right]_a^b$$

$$- \frac{4f(\frac{a+b}{2})}{(a-b)^2} \left[ \frac{x^3}{3} - \frac{x^2}{2} (a+b) + abx \right]_a^b$$

$$= \frac{2}{(b-a)^2} \left[ f(b) \left\{ \frac{b^3-a^3}{3} - \frac{(b^2-a^2)}{2} \frac{(3a+b)}{2} + a(\frac{a+b}{2})(b-a) \right\} \right.$$

$$+ f(a) \left\{ \frac{b^3-a^3}{3} - \frac{(b^2-a^2)}{2} (\frac{a+3b}{2}) + b(\frac{a+b}{2})(b-a) \right\}$$

$$\left. - 2f\left(\frac{a+b}{2}\right) \left\{ \frac{b^3-a^3}{3} - \frac{(b^2-a^2)}{2} (a+b) + ab(b-a) \right\} \right]$$

$$= \frac{2}{(b-a)^2} (b-a) \left[ f(b) \left\{ \frac{b^2+a^2+ab}{3} - \frac{(a+b)(3a+4)}{4} + \frac{a^2+ab}{2} \right\} \right.$$

$$+ f(a) \left\{ \frac{b^2+a^2+ab}{3} - \frac{(a+b)(3a+3b)}{4} + \frac{ab+b^2}{2} \right\}$$

$$\left. - 2f\left(\frac{a+b}{2}\right) \left\{ \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{2} + ab \right\} \right]$$

$$\begin{aligned}
&= \frac{2}{b-a} \left[ f(b) \left\{ \frac{4a^2 + 4b^2 + 4ab - 9a^2 - 12ab - 3b^2 + ca^2 + cab}{12} \right\} \right. \\
&\quad + f(a) \left\{ \frac{4a^2 + 4b^2 + 4ab - 3a^2 - 12ab - 9b^2 + cab + cb^2}{12} \right\} \\
&\quad \left. - 2f\left(\frac{a+b}{2}\right) \left\{ \frac{2a^2 + 2b^2 + 2ab - 3a^2 - 3b^2 - cab + cab}{6} \right\} \right] \\
&= \frac{2}{b-a} \left[ f(b) \left( \frac{a^2 + b^2 - 2ab}{12} \right) + f(a) \left( \frac{a^2 + b^2 - 2ab}{12} \right) \right. \\
&\quad \left. - 2f\left(\frac{a+b}{2}\right) \left( \frac{-a^2 - b^2 + 2ab}{6} \right) \right] \\
&= \frac{2}{b-a} \left[ f(b) \frac{(a-b)^2}{12} + f(a) \frac{(a-b)^2}{12} + 4f\left(\frac{a+b}{2}\right) \frac{(a-b)^2}{12} \right] \\
&= \frac{2}{b-a} \cdot \frac{(b-a)^2}{12} \left( f(b) + f(a) + 4f\left(\frac{a+b}{2}\right) \right) \\
&= \frac{(b-a)}{c} \left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)
\end{aligned}$$