Worksheet 1 (contd.)

$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (1,1)$

1b) $\operatorname{cvx}(v_1, v_2, v_3)$

Any cvx. comb. of v_1, v_2, v_3: $\quad t_1 v_1 + t_2 v_2 + t_3 v_3$, $\quad$ (*) $t_1 + t_2 + t_3 = 1$, $\quad$ (**) $t_1, t_2, t_3 \geq 0$

$t_1 = a$, $t_2 = b$

(*) $\Rightarrow t_3 = 1 - a - b$.

(**) $\boxed{a \geq 0,\ b \geq 0 \quad \boxed{1-a-b \geq 0} \quad \Rightarrow \boxed{a+b \leq 1}}$

$= a v_1 + b v_2 + (1-a-b) v_3$

$= a(1,0) + b(0,1) + (1-a-b)(1,1)$

$= (\cancel{a} + 1 - \cancel{a} - b,\ \cancel{b} + 1 - a - \cancel{b})$

$= (\underline{1-b},\ \underline{1-a})$ where

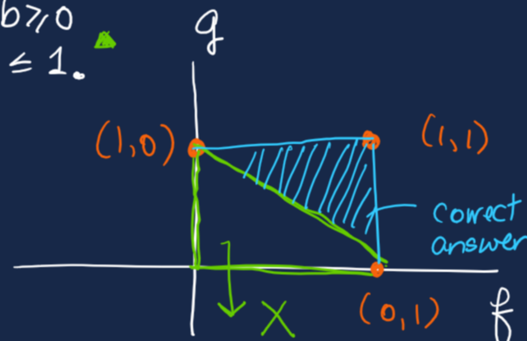
$a \geq 0,\ b \geq 0$

$a + b \leq 1.$

Set $f = 1-b$, $g = 1-a$

where $f \leq 1$; $g \leq 1$; $\cancel{f+g \leq 1}$

$1 \leq f+g \leq 2$



All (f,g) s.t. $f \leq 1,\ g \leq 1,\ f+g \leq 1$

$f \geq 0,\ g \geq 0$

(*)
$0 \leq a+b \leq 1$

$0 \leq 2 - (f+g) \leq 1$

$\Downarrow$

$f+g \leq 2$

$\rightarrow$

$0 \leq 2 - (f+g) \leq 1$

$\Downarrow$

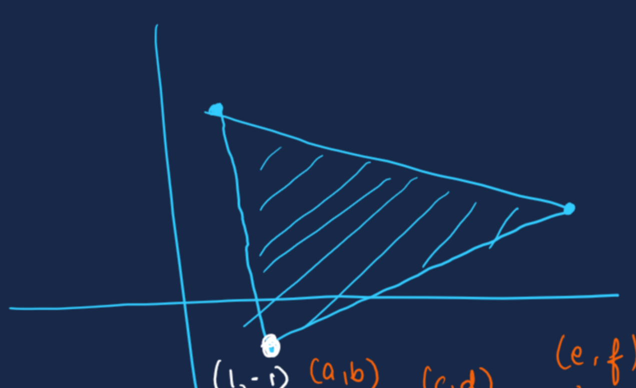
$f+g \geq 1$

Claim. Each v_j is always in $\operatorname{cvx}(v_1, \dots, v_k)$.

Proof: Write

$v_j = 0 \cdot v_1 + \dots + 1 \cdot v_j + \dots 0 v_k$ ∎

Part f)



$= \quad =$

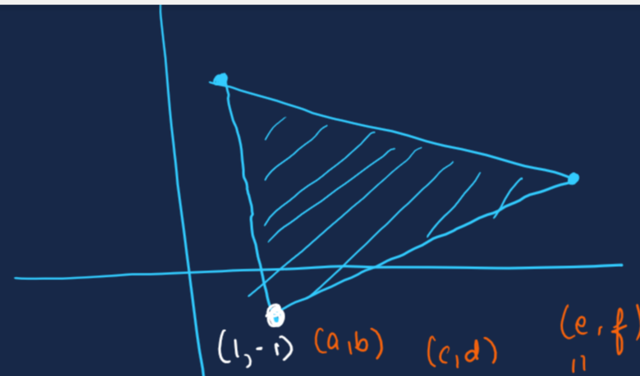
$t_1 v_1 + t_2 v_2 + t_3 v_3$

$t_1, t_2, t_3 \geq 0$

? $\Leftarrow$

No, picture in the

because



$$\left\{ t_1 v_1 + t_2 v_2 + t_3 v_3 \;\middle|\; t_1, t_2, t_3 \geq 0 \right\}$$

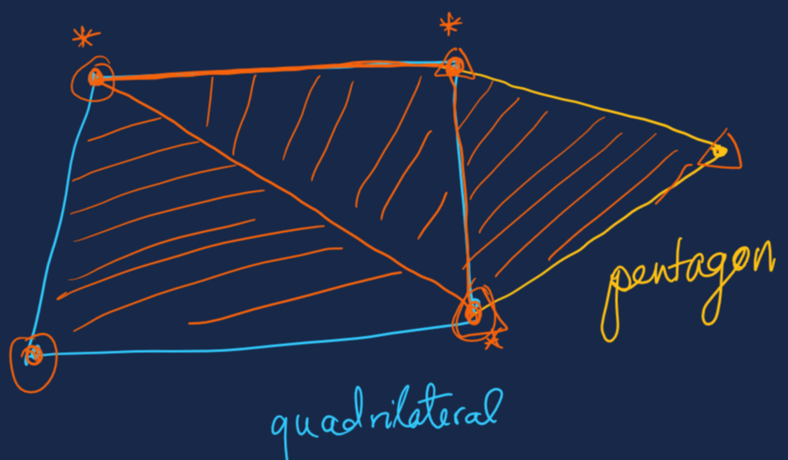
?

No, picture in the 1st quadrant?

because vectors themselves could have negative components

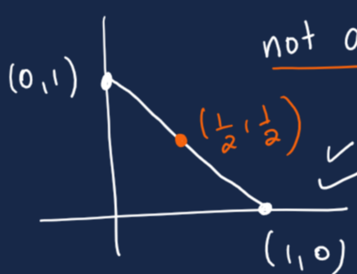
$$\left(\underbrace{t_1}_{\geq 0} a + \underbrace{t_2}_{\geq 0} c + \underbrace{t_3}_{\geq 0} e,\; \underbrace{t_1}_{\geq 0} b + \underbrace{t_2}_{\geq 0} d + \underbrace{t_3}_{\geq 0} f\right)$$

Subhashree's Guess: cvx hull of k points : polygon with those points as vertices.

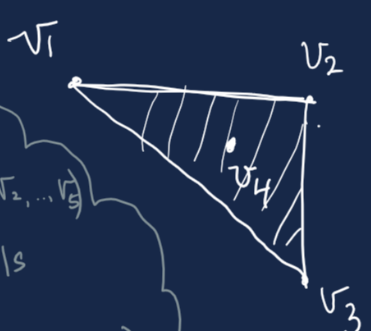


k = 3 "typical"

not a triangle!



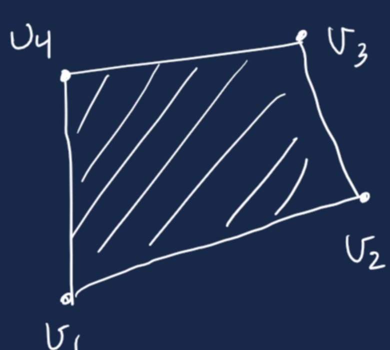
k = 4



cvx$(v_1, v_2, \ldots, v_5)$

v/s

cvx(v_1, v_2, v_3)



$cvx(v_1, v_2, \dots, v_5)$

v/s

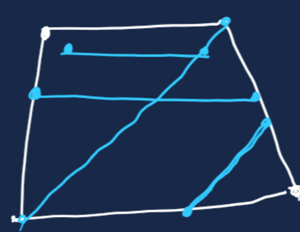
$cvx(v_1, v_2, v_3)$

$k=5$

Answer f) The convex hull of k vectors in $\mathbb{R}^2$ is a $\boxed{\text{convex}}$ polygon with $\boxed{\text{at most}}$ k vertices / sides.

nonconvex polygon

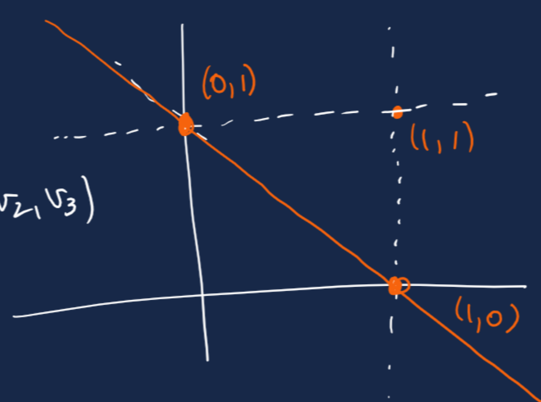
Convex polygon



line segment joining any pair of points is within the shape.

1b) $aff(v_1, v_2, v_3)$

$v_1 = (1,0)$
$v_2 = (0,1)$
$v_3 = (1,1)$



Subhashree's answer:

$x = 1, \; y = 1, \; x + y = 1 \subset aff(v_1, v_2, v_3)$

any affine combination:

$$t_1 v_1 + t_2 v_2 + t_3 v_3$$

$$\|$$

$$t_1 v_1 + t_2 v_2 + (1 - t_1 - t_2) v_3$$

Idea: take alternately

- $t_1 = 0 \Rightarrow t_2 + t_3 = 1$
- $t_2 = 0 \Rightarrow t_1 + t_3 = 1$
- $t_3 = 0 \Rightarrow t_1 + t_2 = 1$

$$= t_1(1,0) + t_2(0,1) + (1-t_1-t_2)(1,1) \quad \text{where } t_1, t_2 \in \mathbb{R}$$

$$= (1-t_2, 1-t_1), \quad \text{here } t_1, t_2 \in \mathbb{R}.$$

$$= (a,b) \rightsquigarrow \quad t_2 = 1-a, \quad t_1 = 1-b$$

$$\Rightarrow t_3 = 1-t_1-t_2.$$

Answer: $\mathrm{aff}(v_1, v_2, v_3) = \mathbb{R}^2$.

Definition: A $\boxed{\text{convex polytope}}$ in $\mathbb{R}^n$ is the convex hull of finitely many points.

Polygon is a polytope in $\mathbb{R}^2$

PROBLEM 1, PART C.

~~$\mathrm{lin}(v_1 - v_2) + v_2$ vts~~

~~$\mathrm{lin}(v_1, v_2)$~~

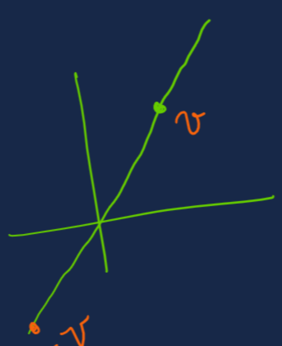
Part c could also be:

$$\mathrm{aff}(v_1, v_2) = \mathrm{lin}(v_1 - v_2) + v_1$$

Can it be that

$$\| ?$$

$$\mathrm{aff}(v_2, v_1) = \mathrm{lin}(v_2 - v_1) + v_1$$



Note: $\mathrm{lin}(v) = \mathrm{lin}(-v)$

HOW TO VISUALIZE IN 4D

3d

4th dimension as time

$t = \varepsilon$

$t = 1$

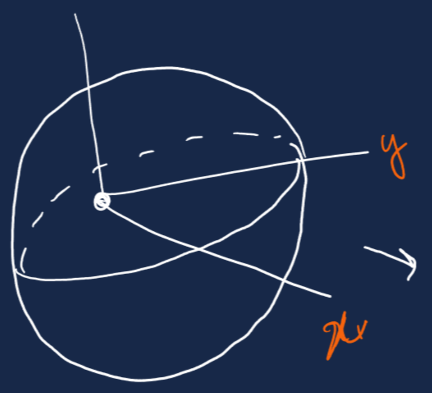
HOW TO VISUALIZE IN 4D

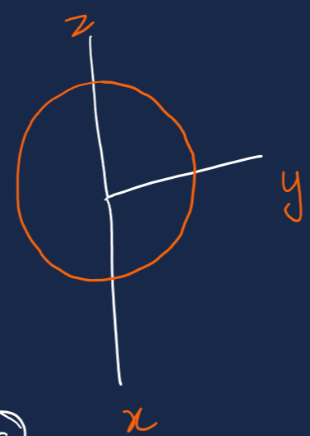
3d

4th dimension as time.

$t = \varepsilon$

$t=0$ slice





$t=1$

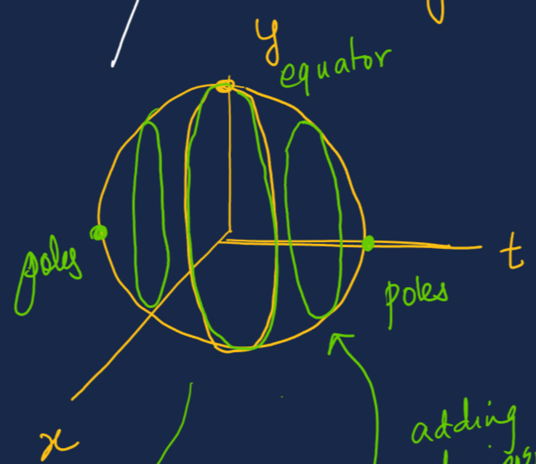
$$x^2+y^2+z^2+t^2=1 \qquad \boxed{t=0}$$

$$x^2+y^2+z^2 = 1-\varepsilon^2 \qquad \boxed{t=\varepsilon}$$

$$x^2+y^2+z^2=0 \qquad \boxed{t=1}$$

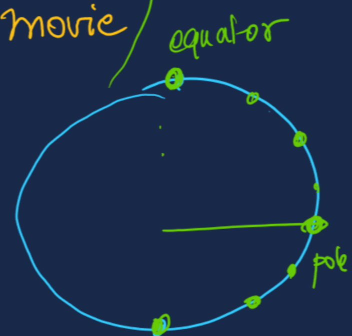
$$x^2+y^2+z^2<0 \qquad t>1$$

$x^2+y^2+t^2=1$



adding dimension

movie



FLATLAND