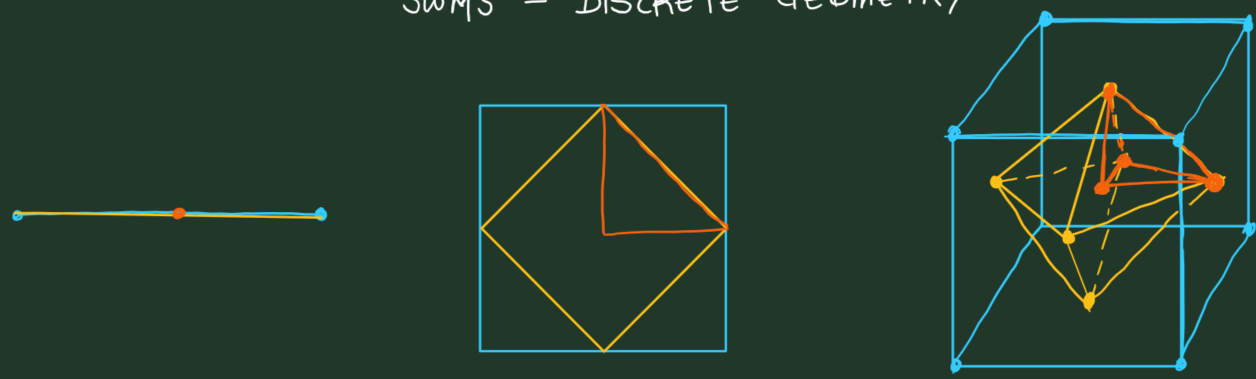
June 22, 2021

SWMS — DISCRETE GEOMETRY



LAST TIME: A convex polytope in $\mathbb{R}^n$ is the convex hull of a finite collection of points in $\mathbb{R}^n$.

Examples: 1. The standard (hyper)cube in $\mathbb{R}^n$: $\overbrace{[-1,1]\times\cdots\times[-1,1]}^{n\text{-times}}$

= { convex hull of all those points in $\mathbb{R}^n$ whose coordinates are either $+1$ or -1. Q. How many such points are there?

2 : # of choices

A1. $2n$?

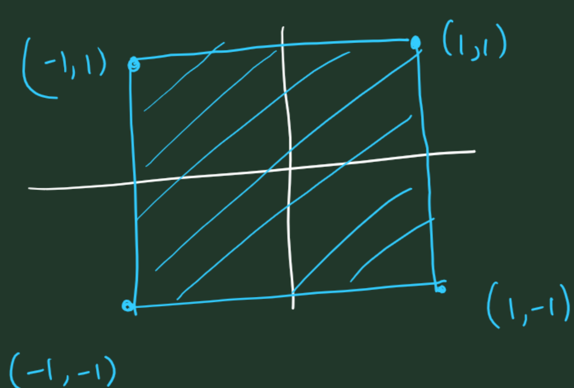
A2. 2^n? ✓

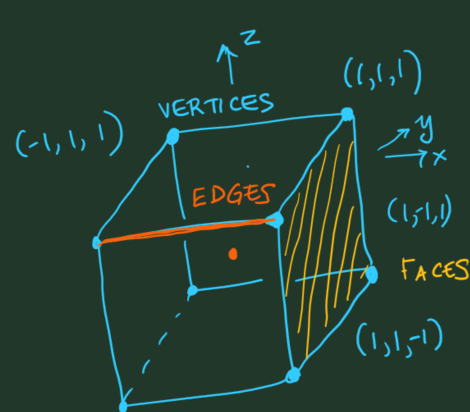
A3. n^2?

$$\underbrace{2\times 2\times 2\cdots\times 2}_{n\text{-coordinates}} = 2^n$$

$$= \left\{ x = (x_1,\dots,x_n) \in \mathbb{R}^n : \max\{|x_1|,\dots,|x_n|\} \le 1 \right\}.$$

$$= \left\{ x = (x_1,\dots,x_n) \in \mathbb{R}^n : \begin{array}{c} -1 \le x_1 \le 1 \\ \vdots \\ -1 \le x_n \le 1 \end{array} \right\}$$





#V: 8

#E: 12

#F: 6

2n points

2. Cross polytope K_n in $\mathbb{R}^n$: $\mathrm{cvx}(\overbrace{\pm e_1, \dots, \pm e_n}^{2n \text{ points}})$, where

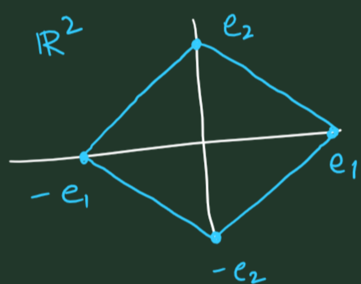
$e_j = (0, \dots, 1, \dots, 0)$.
↑ j^{th} position

Note: $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

"proof" $\equiv \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_1| + \dots + |x_n| \le 1 \}$

K_1

$\mathbb{R}$



$\mathbb{R}^2$

cross polytopes

square pyramid

\# V : 6
\# E : 12
\# F : 8

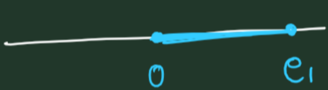
(octahedron)

3. The standard $\boxed{\text{simplex}}$ S_n in $\mathbb{R}^n$ $= \mathrm{cvx}(0, e_1, \dots, e_n)$.

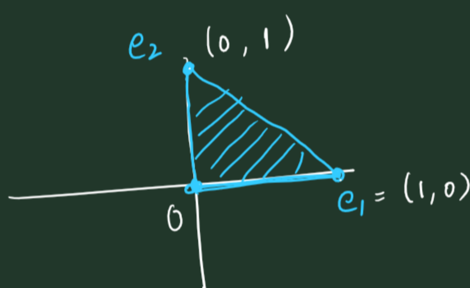
" the simplest n-dim polytope in $\mathbb{R}^n$."

(tetrahedron)

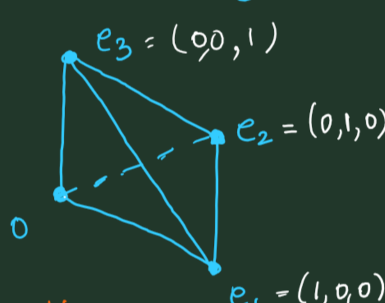
S_1



S_2



S_3



\# V : 4
\# E : 6
\# F : 4.

Observe : $V - E + F = 2$

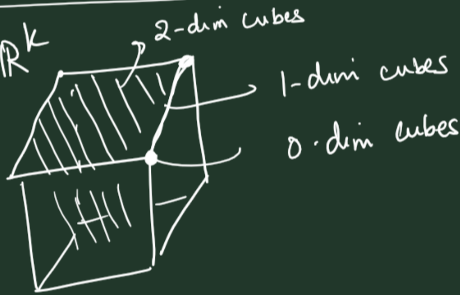
(topology)

General fact about convex polytopes in $\mathbb{R}^3$: $V - E + F = 2$.

$C_k \subset \mathbb{R}^k$

Guess : The boundary of the 4-dim hyper cube ~~set~~ consists

$C_k \subset \mathbb{R}^k$



Guess: The boundary of the 4-dim hyper cube ~~set~~ consists of C_1 as edges
C_2 as faces (2)
C_3 as 3-d faces.

Worksheet 2

Solution space of a homogeneous system $\vec{x}$

$A\vec{x} = 0 \;=\; A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

a) Null space of $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$.

$N = \{ c(-2,5,1) : c \in \mathbb{R} \}$.

$= \text{lin}((-2,5,1))$.

$\text{lin}(v) = \begin{cases} \text{line joining } 0 \,\&\, v, & \text{if } v \neq 0 \\ \{0\}, & v = 0. \end{cases}$

General: null space is always a linear span / subspace.

b) Solution space of

$$x + 2z = 1$$
$$3x + y + z = 2.$$

$\underline{A}\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Inhomogeneous system.

$\underline{S} = \{ \overbrace{(1,-1,0)}^{w_0} + c(-2,5,1) : c \in \mathbb{R} \}$.

$=$ particular solution + null space of A

$= \underline{w_0} + \underline{\text{lin}(\overbrace{(-2,5,1)}^{w_1})}$.

First: $0 \in \text{lin}(v_1, \dots, v_k)$
why: $t_1 = \dots = t_k = 0$.

Note: $0 \notin S$

So, S is not a linear span.

Affine span?

$= \left\{ \boxed{1 \cdot w_0 - c w_0} + c w_1 : c \in \mathbb{R} \right\}$
$+ c w_0$

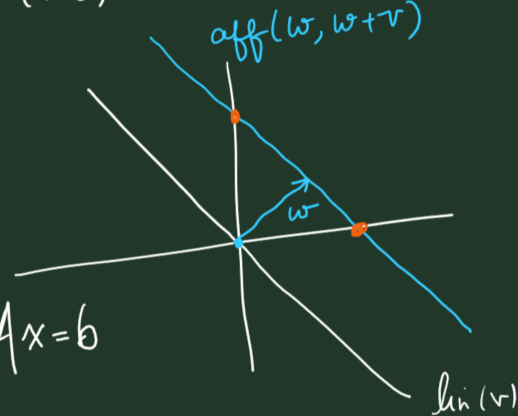
$= \left\{ \overset{t_1}{(1-c)} w_0 + \overset{t_2}{c}(w_0 + w_1) : c \in \mathbb{R} \right\}$ $t_1 + t_2 = 1$.

$$= \operatorname{aff}(w_0, w_0+w_1).$$

$$v_2 + \operatorname{lin}(v_1 - v_2) = \operatorname{aff}(v_1, v_2)$$

Summary: $w_0 + \operatorname{lin}(w_1) = \operatorname{aff}(w_0, w_0+w_1).$

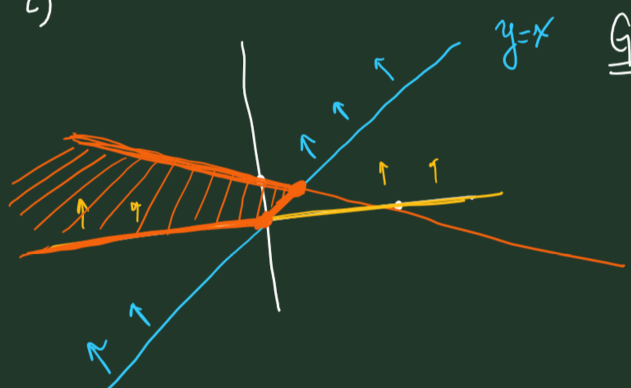
Problem 1 part c).



General fact: Solution space of $Ax=b$ is always an <u>affine hull</u>.

HW. What about range of A?

c)



Guess: affine hull of which vectors?

$$\begin{aligned} x - y &\le 0 \\ -y &\le 0 \\ x + 2y &\le 1 \end{aligned} \qquad \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

component-wise

$A\vec{x} = b$

Linear algebra

$A\vec{x} \le b$ } Linear programming